ON A NUMERICAL METHOD FOR DETERMINING THE ROOTS OF CHARACTERISTIC EQUATIONS

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Determination of real roots. Suppose we are given the equation

$$x^n + A_{n-1}x^{n-1} + \ldots + A_1x + A_0 = 0 \tag{1}$$

If Equation (1) has a real root λ then this equation can be written in the form

$$(x - \lambda)(x^{n-1} + a_{n-2}x^{n-2} + a_{n-3}x^{n-3} + \ldots + a_1x + a_0) = 0 \tag{2}$$

Let us determine the coefficients a_i in such a way that Equation (2) may differ from (1) only by a term not containing x. This can easily be accomplished by solving the system of equations

$$\begin{aligned} a_{n-2} &= \lambda + A_{n-1} \\ a_{n-3} &= \lambda a_{n-2} + A_{n-2} \\ a_{n-4} &= \lambda a_{n-3} + A_{n-3} \\ &\ldots\ldots\ldots\ldots \\ a_0 &= \lambda a_1 + A_1 \end{aligned} \tag{3}$$

The product $A_0^{\,i} = -a_0\lambda_i$ yields the term not containing x, and is, generally speaking, different from A_0.

Having two values A_0' and A_0'' which correspond to two arbitrarily chosen values λ_1 and λ_2 of the root under consideration, we determine a new value λ_3 by interpolation:

$$\lambda_3 = \lambda_1 + \frac{A_0 - A_0'}{A_0' - A_0''}(\lambda_1 - \lambda_2) \tag{4}$$

It is useful to select λ_1 and λ_2 so that $A_0' - A_0$ and $A_0'' - A$ have opposite signs. Having computed A_0^3 for the new value λ_3, we determine the next approximate value λ by interpolation with the values λ_i and λ_j

for which $A_0^{\,i} - A_0$ and $A_0^{\,j} - A_0$ have different signs, and so on.

Determination of complex roots. We first consider the case when the characteristic equation possesses only one pair of complex roots.

Suppose we are given the fourth-degree equation

$$x^4 + A_3x^3 + A_2x^2 + A_1x + A_0 = 0 \tag{5}$$

Let us rewrite this equation in the form

$$(x^2 + a_1x + a_0)\,(x^2 + b_1x + b_0) = 0 \tag{6}$$

and let us determine the coefficients a_i, b_i under the earlier-mentioned condition. This leads to the following equations:

$$\begin{aligned} a_1 + b_1 &= A_3 \\ a_0 + b_0 + a_1b_1 &= A_2 \\ a_0b_1 + a_1b_0 &= A_1 \end{aligned} \tag{7}$$

It is obvious that one of the coefficients, for example a_1, can be assigned arbitrarily.

The value of the term which does not contain x is equal to $A_0 = a_0b_0$. From the two values A_0' and A_0'', which correspond to two values a_1' and a_1'' of the coefficients a_1, we find a new value a_1^{3} by interpolation as follows:

$$a_1^3 = a_1' + \frac{A_0 - A_0'}{A_0' - A_0''}\,(a_1' - a_1'') \tag{8}$$

Repeating this process, we find $a_0^{\,i}$ for which $A_0^{\,i}$ differs but little from A_0, and we thus determine an approximate value of the sought solution.

Let us consider the equation which has n pairs of complex roots:

$$x^{2n} + A_{2n-1}x^{2n-1} + A_{2n-x}x^{2n-2} + \ldots + A_1x + A_0 = 0 \tag{9}$$

We rewrite this equation in the following form:

$$(x^2 + a_1x + a_0)\,(x^{2n-2} + b_{2n-3}x^{2n-3} + \ldots + b_1x + b_0) = 0 \tag{10}$$

Having been given the values of the coefficients a_1 and a_0, we determine the coefficients b_{2n-3}, b_{2n-4}, ..., b_0 by requiring that, after the multiplication of the two factors in (10), the coefficients of the resulting equation be the same as the corresponding coefficients of Equation (9) except for the last two terms which may differ from A_1 and A_0.

The necessary conditions on the coefficients are given by the equations

$$
\begin{gathered}
a_1 + b_{2n-3} = A_{2n-1} \\
b_{2n-4} + a_1 b_{2n-3} + a_0 = A_{2n-2} \\
b_{2n-5} + a_1 b_{2n-4} + a_0 b_{2n-3} = A_{2n-3} \\
b_{2n-6} + a_1 b_{2n-5} + a_0 b_{2n-4} = A_{2n-4} \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
b_0 + a_1 b_1 + a_0 b_2 = A_2 \\
a_1 b_0 + a_0 b_1 = A_1 \\
a_0 b_0 = A_0
\end{gathered}
\tag{11}
$$

Fixing a_0 and varying a_1, we use the earlier-given method to determine the value of a_1 so that the coefficients A_1' of x in Equation (11) may differ little from the coefficient A_1.

Repeating this procedure for different values of a_0 we determine for what values of a_1 and a_0 the coefficients of Equations (9) and (10) will be nearly equal. In this manner we find approximate values for the first pair of complex roots. We thus obtain a new equation whose degree is less, by two, than the degree of the original equation.

The construction of the curves $a_0'(\lambda)$, $A_0'(a_1)$ and $A_0'(a_1, a_0)$, $A_1'(a_1, a_0)$ is helpful in finding the roots.

The method presented requires only quite simple computations; it is routine and easily adapted for programming on computing machines.

Translated by H.P.T.